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Problems BAB I Analisis Vektor

Problem 1.11

Find the gradien of the following functions :

a) $f(x,y,z) = x^2 + y^3 + z^4$

b) $f(x,y,z) = x^2 y^3 z^4$

c) $f(x,y,z) = e^x \sin(y) \ln(z)$

Solution

a) $f(x,y,z) = x^2 + y^3 + z^4$

$\vec{\nabla} f = 2x\hat{i} + 3y^2\hat{j} + 4z^3\hat{k}$

b) $f(x,y,z) = x^2 y^3 z^4$

$\vec{\nabla} f = 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^4\,\hat{k}$

c) $f(x,y,z) = e^x \sin(y) \ln(z)$

$\vec{\nabla} f = e^x \sin(y) \ln(z)\,\hat{i} + e^x \cos(y) \ln(z)\,\hat{j} +$
$e^x \sin(y) (1/z)\,\hat{k}$

Problem 1.12

A height of a certain hill (in feet) is given by $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$, where is the y is distance (in miles) north, x the distance east of south hadley.

a) Where is the top of the hill located ?

b) How high is the hill ?

c) How steep is the slope (in feet per mile) at a point 1 miles north and 1 mile east of south hadley? In what direction is the slope steepest at that point ?

Solution

a) Dimanakah letak puncak bukit tersebut ?

⇒ Tentukan fungsi gradien

$\vec{\nabla} h = 10\,[(2y - 6x - 18)\,\hat{i} + (2x - 8y + 28)\,\hat{j}]$

untuk menentukan puncak bukit, gunakan syarat $\vec{\nabla} b = 0$ (puncak bukit merupakan salah satu jenis stasioner)

$\vec{\nabla} b = 10 \left[(2y - 6x - 18)\hat{i} + (2x - 8y + 28)\hat{j} \right] = 0$, menghasilkan sistem persamaan linear dua peubah

$$\left.\begin{aligned} 2y - 6x - 18 &= 0 \\ 2x - 8y + 28 &= 0 \end{aligned}\right\} \text{ Solusi dari sistem pertemuan } \ln = (x, y) = (-2, 3)$$

Dengan demikian, puncak bukit tersebut berada pada 2 km sebelah barat dan 3 km utara Bandung.

b) Berapa ketinggian bukit tersebut ?

Substitusikan $(x, y) = (-2, 3)$ pada $b(x, y)$

$b(-2, 3) = 10 \; (-12 - 12 - 36 + 36 + 84 + 12) = 720$ m

Jadi, ketinggian bukit tersebut adalah 720 m

c) Seberapa curam kemiringan (dalam satuan m/km) pada sebuah titik 1 km utara dan 1 km timur kota Bandung ? pada arah manakah kemiringan tercuram di titik tersebut ?

Substitusikan $(x, y) = (1, 1)$ pada $\vec{\nabla} b$

$\vec{\nabla} b \, (1,1) = 10 \left[(2 - 6 - 18)\hat{i} + (2 - 8 + 28)\hat{j} \right] = 220 \, (-\hat{i} + \hat{j})$

$|\vec{\nabla} b| = 220\sqrt{2} \approx 311$ m/km, arahnya ke barat laut (135° dari sumbu x positif)

$\boxed{\text{Problem 1.13}}$

Let $\mathscr{r}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and let $\mathscr{r}$ be its length. Show that

a) $\nabla (\mathscr{r}^2) = 2\mathscr{r}$

b) $\nabla (1/\mathscr{r}) = -\mathscr{r}/\mathscr{r}^2$

c) What is the general formula for $\nabla(\mathscr{r}^n)$

Solution

a) $\nabla (\mathscr{r}^2) = 2\mathscr{r}$

$$\mathscr{r} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

$$\mathscr{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

$$\nabla(\mathscr{r}^2) = \frac{\partial}{\partial x} \left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)\hat{x} + \left((x - x')^2 + (y - y')^2 + (z - z')^2 \right)\hat{y} +$$

$$\frac{\partial}{\partial z}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)$$

$$= 2\,(x-x')\,\hat{x} + 2\,(y-y')\,\hat{y} + 2\,(z-z')\,\hat{z}$$

$$= 2\,\eta$$

b) $\nabla\left(\frac{1}{\eta}\right) = -\frac{\hat{\eta}}{\eta^2}$

$$= \frac{\partial}{\partial x}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)^{-\frac{1}{2}}\hat{z}$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)^{1/2}\right]^{-\frac{3}{2}} 2\,(x-x')\,\hat{x} -$$

$$\frac{1}{2}\left[\frac{\partial}{\partial y}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)^{-\frac{1}{2}}\right]^{-\frac{3}{2}} 2\,(y-y')\,\hat{y} -$$

$$\frac{1}{2}\left[\frac{\partial}{\partial z}\left(\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)^{-\frac{1}{2}}\right]^{-\frac{3}{2}} 2\,(z-z')\,\hat{z}$$

$$= -\left(\frac{2}{2}\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right)^{-\frac{1}{2}}\Big]^{-\frac{3}{2}}\Big((x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}\Big)$$

$$= -\frac{1}{\eta^3}\,\eta$$

$$= -\frac{1}{\eta^2}\,\hat{\eta}$$

$$= -\frac{\hat{\eta}}{\eta^2}$$

c) What is the general formula

$$\frac{\partial}{\partial x}\left(\eta^n\right) = n\,\eta^{n-1}\,\frac{\partial \eta}{\partial x}$$

$$= n\,\eta^{n-1}\left(\frac{1}{2}\,\frac{1}{\eta}\;2\,\eta\,x\right)$$

$$= n\,\eta^{n-1}\,\hat{\eta}_x$$

Problem 1.14

Suppose that f is a function of two variables (y and z) only. Show that the gradien $\nabla f = (\partial f / \partial y)\hat{y} + (\partial f / \partial z)\hat{z}$ transforms as a vector under rotations, eq. 129 [Hint : $(\partial f / \partial \bar{y}) = (\partial f / \partial y)(\partial y / \partial \bar{y}) + (\partial f / \partial z)(\partial z / \partial \bar{y})$, and the anologous formula for $\partial f / \partial \bar{z}$. We know that $\bar{y} = y \cos\theta + z \sin\theta$ and $\bar{z} = -y \sin\theta + i \cos\theta$. Solve these equations for y and z (as functions of $\bar{y}$ and $\bar{z}$), and compute the needed derivatives $\partial y / \partial \bar{y}$, $\partial z / \partial \bar{y}$, etc.)

Solution

$\bar{y} = y \cos\theta + z \sin\theta$

$\bar{y} \sin\theta = y \sin\theta \cos\theta + z \sin^2\theta$

$\bar{z} = -y \sin\theta + z \cos\theta$

$\bar{z} \cos\theta = -y \sin\theta \cos\theta + z \cos^2\theta$

$\bar{y} \sin\theta + \bar{z} \cos\theta = z(\sin^2\theta + \cos^2\theta) = z$

Sehingga, $\dfrac{\partial y}{\partial y} = \cos\theta$, $\dfrac{\partial y}{\partial z} = -\sin\theta$; $\sin\theta \; \dfrac{\partial z}{\partial z} = \cos\theta$

$(\nabla f)y = \dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial y} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial y} = \cos\theta \, (\nabla f)y + \sin\theta \, (\nabla f)z$

$(\nabla f)z = \dfrac{\partial f}{\partial z} = \dfrac{\partial f}{\partial z}\dfrac{\partial F}{\partial z} \; \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial z} = -\sin\theta \, (\nabla f)y \; \cos\theta \, (\nabla f)z$