Latihan Soal Deret Fourier (2)

Oleh:
AMIR SUPRIYANTO
FISIKA FMIPA UNILA

1. Tentukan deret Fourier dari

$$f(x) = 2 - x; \quad -2 < x < 2$$

Penyelesaian

Menentukan koefisien:

$$a_0 = \frac{1}{2}\int_{-2}^{2}(2-x)dx = \frac{1}{2}\left\{2x - \frac{1}{2}x^2\Big|_{-2}^{2}\right\}$$

$$= \frac{1}{2}\left\{2(2-(-2) - \frac{1}{2}\left(2^2 - (-2)^2\right)\right\}$$

$$= \frac{1}{2}\left\{2(4) - \frac{1}{2}(4-4)\right\} = 4$$

$$a_n = \frac{1}{2}\int_{-2}^{2}(2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{-2}^{2}\cos\left(\frac{n\pi x}{2}\right)dx - \frac{1}{2}\int_{-2}^{2}x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2} - \frac{1}{2}\,\frac{2}{n\pi}\int_{-2}^{2}x\,d\sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{2}{n\pi}\left\{\sin\left(\frac{n\pi 2}{2}\right) - \sin\left(\frac{n\pi(-2)}{2}\right)\right\}$$

$$- \frac{1}{n\pi}\left\{x\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2} - \int_{-2}^{2}\sin\left(\frac{n\pi x}{2}\right)dx\right\}$$

$$
\begin{aligned}
a_n &= \frac{2}{n\pi}\{\sin(n\pi) - \sin(-n\pi)\} \\
&\quad - \frac{1}{n\pi}\left\{2\sin\left(\frac{n\pi 2}{2}\right) - (-2)\sin\left(\frac{n\pi(-2)}{2}\right)\right. \\
&\quad \left. + \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2}\right\} \\
&= \frac{2}{n\pi}\{0 - 0\} \\
&\quad - \frac{1}{n\pi}\left\{2\sin(n\pi) + 2\sin(-n\pi)\right. \\
&\quad \left. + \frac{2}{n\pi}\left[\cos\left(\frac{n\pi 2}{2}\right) - \mathrm{cox}\left(\frac{n\pi(-2)}{2}\right)\right]\right\} \\
&= -\frac{1}{n\pi}\left\{0 + 0 + \frac{2}{n\pi}[\cos(n\pi) - \cos(-n\pi)]\right\} \\
&= -\frac{1}{n\pi}\left\{\frac{2}{n\pi}[\cos(n\pi) - \cos(n\pi)]\right\} = 0
\end{aligned}
$$

$$b_n = \frac{1}{2}\int_{-2}^{2}(2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{-2}^{2}\sin\left(\frac{n\pi x}{2}\right)dx - \frac{1}{2}\int_{-2}^{2}x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\cos\ \left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2} + \frac{1}{2}\,\frac{2}{n\pi}\int_{-2}^{2}x\,d\cos\left(\frac{n\pi x}{2}\right)$$

$$= -\frac{2}{n\pi}\left\{\cos\ \left(\frac{n\pi 2}{2}\right) - \cos\ \left(\frac{n\pi(-2)}{2}\right)\right\}$$

$$+\ \frac{1}{n\pi}\left\{x\ \cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2} - \int_{-2}^{2}\cos\left(\frac{n\pi x}{2}\right)dx\right\}$$

$$
\begin{aligned}
b_n = -\frac{2}{n\pi}\{\cos(n\pi) - \cos(-n\pi)\} \\
&+ \frac{1}{n\pi}\left\{2\cos\left(\frac{n\pi 2}{2}\right) - (-2)\cos\left(\frac{n\pi(-2)}{2}\right)\right. \\
&\left. - \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{2}\right\} \\
&= -\frac{2}{n\pi}\{\cos(n\pi) - \cos(n\pi)\} \\
&+ \frac{1}{n\pi}\left\{2\cos(n\pi) + 2\cos(n\pi)\right. \\
&\left. - \frac{2}{n\pi}\left[\sin\left(\frac{n\pi 2}{2}\right) - \sin\left(\frac{n\pi(-2)}{2}\right)\right]\right\} \\
&= \frac{1}{n\pi}\left\{4\cos(n\pi) - \frac{2}{n\pi}[\sin(n\pi) - \sin(-n\pi)]\right\} \\
&= \frac{1}{n\pi}\left\{4\cos(n\pi) - \frac{2}{n\pi}[0-0]\right\} = \frac{4\cos(n\pi)}{n\pi} = \frac{4(-1)^n}{n\pi}
\end{aligned}
$$

- $a_0 = 4$
- $a_n = 0$
- $b_n = \frac{4(-1)^n}{n\pi}$

$$f(x) = 2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

2. Tentukan deret Fourier dari

$$f(x) = x^2; \quad -\frac{1}{2} < x < \frac{1}{2}$$

$f(x) = x^2$; adalah fungsi genap

Penyelesaian

Menentukan koefisien:

$$a_0 = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{2}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{2}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right) = \frac{2}{3}\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{12}$$

$$a_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\cos(2n\pi x)\,dx = \frac{2}{2n\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 d\sin(2n\pi x)$$

$$= \frac{1}{n\pi}\left\{x^2\sin(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2n\pi x)\ \ 2x\,dx\right\}$$

$$= \frac{1}{n\pi}\left\{\left(\frac{1}{2}\right)^2 \sin\left(2n\pi\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2 \sin\left(-2n\pi\frac{1}{2}\right)\right.$$

$$\left. + \frac{1}{2n\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} x\,d\cos(2n\pi x)\right\}$$

$$a_n = \frac{1}{n\pi}\left\{\frac{1}{4}\sin(n\pi) - \frac{1}{4}\sin(-n\pi)\right.$$

$$\left. + \frac{1}{2n\pi}\left[x\cos(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2n\pi x)\ dx \right]\right\}$$

$$= \frac{1}{n\pi}\left\{0 - 0 + \frac{1}{2n\pi}\left[\frac{1}{2}\cos(n\pi) + \frac{1}{2}\cos(-n\pi) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2n\pi x)\ dx \right]\right\}$$

$$= \frac{1}{n\pi}\left\{\frac{1}{2n\pi}\left[\cos(n\pi) - \frac{1}{2n\pi}\sin(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}}\right]\right\}$$

$$= \frac{1}{n\pi}\left\{\frac{1}{2n\pi}\left[\cos(n\pi) - \frac{1}{2n\pi}\left(\sin(n\pi) - \sin(-n\pi)\right)\right]\right\} = \frac{\cos(n\pi)}{2n^2\pi^2}$$

$$= \frac{(-1)^n}{2n^2\pi^2}$$

$$b_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \sin(2n\pi x)\,dx = -\frac{2}{2n\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 d\cos(2n\pi x)$$

$$= -\frac{1}{n\pi}\left\{ x^2\cos(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2n\pi x)\;\, 2x\,dx \right\}$$

$$= -\frac{1}{n\pi}\left\{ \left(\frac{1}{2}\right)^2 \cos\left(2n\pi\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2 \cos\left(-2n\pi\frac{1}{2}\right) \right.$$

$$\left. + \frac{1}{2n\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} x\,d\sin(2n\pi x) \right\}$$

$$b_n = -\frac{1}{n\pi}\left\{\frac{1}{4}\cos(n\pi) - \frac{1}{4}\cos(-n\pi) + \frac{1}{2n\pi}\left[x\sin(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2n\pi x)\ dx\right]\right\}$$

$$= -\frac{1}{n\pi}\left\{0 + \frac{1}{2n\pi}\left[\frac{1}{2}\sin(n\pi) + \frac{1}{2}\sin(-n\pi) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2n\pi x)\ dx\right]\right\}$$

$$= -\frac{1}{n\pi}\left\{\frac{1}{2n\pi}\left[\frac{1}{2n\pi}\cos(2n\pi x)\Big|_{-\frac{1}{2}}^{\frac{1}{2}}\right]\right\}$$

$$= -\frac{1}{n\pi}\left\{\frac{1}{2n\pi}\left[\frac{1}{2n\pi}(\cos(n\pi) - \cos(-n\pi))\right]\right\} = 0$$

Dari soal tampak bahwa fungsi $f(x) = x^2$ adalah fungsi genap, sehingga: $b_n = 0$.

Sebaliknya bila $f(x) =$ fungsi ganjil maka $a_n = 0$

Deret Fourier:

$$f(x) = \frac{1}{24} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2\pi^2} \cos\ (2n\pi x)$$

3. Tentukan deret Fourier dari

$$f(x) = x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$f(x) = x$; adalah fungsi ganjil

Penyelesaian

Menentukan koefisien:

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx = \frac{2}{\pi}\frac{1}{2}x^2\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\left(\frac{\pi}{2}\right)^2 - \left(-\frac{\pi}{2}\right)^2\right) = 0$$

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nx\,dx = \frac{2}{\pi}.\frac{1}{n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,d\sin nx$$

$$= \frac{2}{n\pi}\left\{x\ \sin nx\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx\ dx\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{\pi}{2}\ \sin\left(\frac{n\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\ \sin\left(-\frac{n\pi}{2}\right) + \frac{1}{n}\ \cos nx\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{\pi}{2}\ \sin\left(\frac{n\pi}{2}\right) - \frac{\pi}{2}\ \sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left. + \frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - \ \cos\left(-\frac{n\pi}{2}\right)\right]\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - \ \cos\left(\frac{n\pi}{2}\right)\right]\right\} = 0$$

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin nx\, dx = -\frac{2}{n\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\, d\cos nx$$

$$= -\frac{1}{n}\left\{x\ \cos nx\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nx\ dx\right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{\pi}{2}\ \cos\left(\frac{n\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\ \cos\left(-\frac{n\pi}{2}\right) - \frac{1}{n}\ \sin nx\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{\pi}{2}\ \cos\left(\frac{n\pi}{2}\right) + \frac{\pi}{2}\ \cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - \ \cos\left(-\frac{n\pi}{2}\right)\right]\right\}$$

$$= -\frac{2}{n\pi}\left\{\pi\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - \ \cos\left(\frac{n\pi}{2}\right)\right]\right\} = -\frac{2\cos\left(\frac{n\pi}{2}\right)}{n}$$

Dari soal tampak bahwa fungsi $f(x) = x$ adalah fungsi ganjil, sehingga: $a_n = 0$.

Sebaliknya bila $f(x) =$ genap maka $b_n = 0$

Deret Fourier:

$$f(x) = -2 \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) \sin(nx)}{n}$$

4. Jika dinyatakan dalam deret Fourier $f(x) = \begin{cases} -1; & l < x < 0 \\ +1; & 0 < x < l \end{cases}$ akan berbentuk: $f(x) = \frac{4}{\pi}\left\{\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) + \cdots\right\}$

Hitunglah bentuk deret: $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Penyelesaian:

$$\overline{|f(x)|^2} = \frac{1}{l}\int_{-l}^{0} dx + \frac{1}{l}\int_{0}^{l} dx = \frac{1}{l}x\Big|_{-l}^{0} + \frac{1}{l}x\Big|_{0}^{l}$$

$$= \frac{1}{l}(0 + l) + \frac{1}{l}(l - 0) = 2$$

$$f(x) = \frac{4}{\pi}\left\{\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{l}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{l}\right) + \cdots\right\}$$

Tampak bahwa: $a_0 = 0;\quad a_n = 0;$ dan $b_n = \frac{4}{\pi(2n-1)}$

Dari Teorema Parseval:

$$\overline{|f(x)|^2} = \left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} {a_n}^2 + \sum_{n=1}^{\infty} {b_n}^2$$

$$= \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$2 = \frac{4^2}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Sehingga:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{2\pi^2}{4^2} = \frac{\pi^2}{8}$$

5. Jika dinyatakan dalam deret Fourier

$$f(x) = x^2; \; -\frac{1}{2} < x < \frac{1}{2}$$

Akan berbentuk:

$$f(x) = \frac{1}{12} - \frac{4}{\pi^2}\left\{\cos(2\pi x) + \frac{1}{2^2}\cos(4\pi x) + \frac{1}{3^2}\cos(6\pi x) + \cdots\right\}$$

Hitunglah bentuk deret: $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Penyelesaian

$$\overline{|f(x)|^2} = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 dx = \frac{2}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right)$$

$$= \frac{2}{3}\left(\frac{1}{8} + \frac{1}{8}\right) = \frac{1}{6}$$

$$\frac{1}{6} = \left(\frac{1}{12}\right)^2 + \frac{4^2}{\pi^4}.\frac{1}{n^4}$$

Tampak bahwa: $a_0 = \frac{1}{12}$; $a_n = \sum_{n=1}^{\infty}\frac{4^2}{\pi^4}.\frac{1}{n^4}$; dan $b_n = 0$

Dari Teorema Parseval:

$$\overline{|f(x)|^2} = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n{}^2 + \frac{1}{2}\sum_{n=1}^{\infty} b_n{}^2$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\frac{1}{6} = \left(\frac{1}{12}\right)^2 + \sum_{n=1}^{\infty} \frac{4^2}{\pi^4}\cdot\frac{1}{n^4}$$

Sehingga:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{4^2}\left\{\frac{1}{6} - \left(\frac{1}{12}\right)^2\right\} = \frac{\pi^4}{4^2}\left\{\frac{24-1}{12^2}\right\} = \frac{23\pi^4}{4^2 12^2}$$