Latihan Soal Deret Fourier

Oleh:

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1. Tentukan Deret fourier dari :

$$f(x) = \begin{cases} 0; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$$

Penyelesaian:

Menentukan Koefisien:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{1}{2}\left. x^2\right|_0^{\pi} = \frac{1}{2\pi}\,\pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\;dx = \frac{1}{n\pi}\int_0^{\pi} x\,d\sin nx = \frac{1}{n\pi}\left[\left.x\ \sin nx\right|_0^{\pi} - \int_0^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{n\pi}\left[\pi\ \sin n\pi - 0 + \frac{1}{n}\left.\cos nx\right|_0^{\pi}\right] = \frac{1}{n\pi}\left[0 + \frac{1}{n}(\cos n\pi - \cos 0)\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$a_1 = \frac{1}{\pi}[-1-1] = -\frac{2}{\pi}$; $a_2 = \frac{1}{2^2\pi}[1-1] = 0$; $a_3 = \frac{1}{3^2\pi}[-1-1] = -\frac{2}{3^2\pi}$;

$a_4 = \frac{1}{4^2\pi}[1-1] = 0$; $a_5 = \frac{1}{5^2\pi}[-1-1] = -\frac{2}{5^2\pi}$;

$a_6 = \frac{1}{6^2\pi}[1-1] = 0$;

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx \ dx = -\frac{1}{n\pi}\int_0^{\pi} x \, d\cos nx$$

$$= -\frac{1}{n\pi}\left[x \ \cos nx\Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx\right]$$

$$= -\frac{1}{n\pi}\left[\pi \left(-1\right)^n - 0 + \frac{1}{n}\sin nx\Big|_0^{\pi}\right]$$

$$= -\frac{1}{n\pi}\left[\pi(-1)^n + \frac{1}{n}(\sin n\pi - \sin 0)\right] = -\frac{(-1)^n}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{\pi(2n-1)^2}\cos nx - \frac{(-1)^n}{n} \sin nx$$

2. Tentukan Deret fourier dari :

$$f(x) = 1 + x;\ -\pi < x < \pi$$

Penyelesaian:

Menentukan Koefisien:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\,dx = \frac{1}{\pi}\left[x + \frac{1}{2}x^2\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi - (-\pi) + \frac{1}{2}(\pi^2 - \pi^2)\right] = 2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 + x) \cos nx \ dx = \frac{1}{\pi} \left[\int_{-\pi}^{\pi} \cos nx \, dx + \int_{-\pi}^{\pi} x \ \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n} \ \sin nx \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} x \, d \sin nx \right]$$

$$= \frac{1}{\pi} \left[\sin n\pi - \sin(-n\pi) + \frac{1}{n} \left\{ x \sin nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin nx \, dx \right\} \right]$$

$$= \frac{1}{n\pi} \left[0 - 0 + \left\{ \pi \sin n\pi + \pi \sin(-n\pi) + \frac{1}{n} \ \cos nx \Big|_{-\pi}^{\pi} \right\} \right]$$

$$= \frac{1}{n\pi} \left\{ \pi 0 + \pi 0 + \frac{1}{n} \ (\cos n\pi - \cos\,(-n\pi)) \right\}$$

$$= \frac{1}{n\pi} \frac{1}{n} \ (\cos n\pi - \cos \ n\pi) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x)\sin nx\ \ dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\sin nx\,dx + \int_{-\pi}^{\pi}x\ \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\ \cos nx\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}x\,d\cos nx\right]$$

$$= -\frac{1}{\pi}\left[\cos n\pi - \cos(-n\pi) + \frac{1}{n}\left\{x\cos nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos nx\,dx\right\}\right]$$

$$= -\frac{1}{n\pi}\left[\ \cos n\pi - \cos n\pi + \ \left\{\pi\cos n\pi + \pi\cos(-n\pi) + \frac{1}{n}\ \sin nx\Big|_{-\pi}^{\pi}\right\}\right]$$

$$= \frac{1}{n\pi}\left\{2\pi\cos n\pi + \frac{1}{n}\ (\sin n\pi - \sin\,(-n\pi))\right\}$$

$$= \frac{1}{n\pi}\left\{2\pi(-1)^n + \frac{1}{n}\ (0-0)\right\} = \frac{2(-1)^n}{n}$$

$$\boldsymbol{f(x) = 1 + \ 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx}$$

3..Tentukan Deret fourier dari :

$$f(x) = \begin{cases} x + \pi; \ -\pi < x < 0 \\ -x; 0 < x < \pi \end{cases}$$

Penyelesaian

Menentukan Koefisien:

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (x+\pi)\, dx - \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{1}{2}x^2 + \pi x\right]\Bigg|_{-\pi}^{0} - \frac{1}{2}x^2\Bigg|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}\pi^2 + \pi^2 - \frac{1}{2}\pi^2\right] = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x+\pi)\cos nx\ \ dx - \int_{0}^{\pi} x\cos nx\ \ dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\ \cos nx\right.$$

$$\left. + \int_{-\pi}^{0} \pi\ \cos nx\, dx - \frac{1}{n}\int_{0}^{\pi} x\, d\sin nx\right]$$

Ada tiga suku berbentuk integral

$$I = \frac{1}{\pi}\int_{-\pi}^{0} x\ \cos nx = \frac{1}{\pi}\frac{1}{n}\int_{-\pi}^{0} x\, d\sin nx$$

$$= \frac{1}{n\pi}\left\{x\ \sin nx\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\sin nx\, dx\right\}$$

$$= \frac{1}{n\pi}\left\{0\ \sin 0 + \pi\sin(-\pi) + \frac{1}{n}\cos nx\Big|_{-\pi}^{0}\right\}$$

$$= \frac{1}{n\pi}\left\{0 - 0 + \frac{1}{n}(\cos 0 - \cos\,(-n\pi))\right\}$$

$$= \frac{1}{n^2\pi}\{1 - (-1)^n\}$$

$$II = \frac{1}{\pi}\int_{-\pi}^{0} \pi \ \cos nx\, dx = -\frac{1}{\pi}\frac{\pi}{n} \ \sin nx\Big|_{-\pi}^{0}$$

$$= -\frac{1}{n\pi}\{ \sin 0 - \sin(-\pi)\} = -\frac{1}{n}(0-0)$$

$$III = -\frac{1}{n\pi}\int_{0}^{\pi} x\, d\sin nx \ = -\frac{1}{n\pi}\left\{x \sin nx\Big|_{0}^{\pi} - \int_{0}^{\pi} \sin nx\, dx\right\}$$

$$= -\frac{1}{n\pi}\left\{\pi \ \sin n\pi - 0 \sin 0 + \frac{1}{n}\cos nx\Big|_{0}^{\pi}\right\}$$

$$= -\frac{1}{n\pi}\left\{0 - 0 + \frac{1}{n}(\cos n\pi - \cos 0)\right\}$$

$$= -\frac{1}{n^2\pi}\{(-1)^n - 1\}$$

Sehingga:

$$a_n = I + II + III$$

$$= \frac{1}{n^2\pi}\{1 - (-1)^n\} + 0$$

$$- \frac{1}{n^2\pi}\{(-1)^n - 1\}$$

$$= \frac{1}{n^2\pi}\{1 - (-1)^n + (-1)^n - 1\} = 0$$

$$b = \frac{1}{\pi}\left[\int_{-\pi}^{0} (x+\pi)\sin nx \ dx - \int_{0}^{\pi} x \sin nx \ dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x \ \sin nx\right.$$

$$\left. + \int_{-\pi}^{0} \pi \ \sin nx \, dx + \frac{1}{n}\int_{0}^{\pi} x \, d\cos nx\right]$$

Ada tiga suku berbentuk integral

$$I = \frac{1}{\pi}\int_{-\pi}^{0} x \ \sin nx = -\frac{1}{\pi}\frac{1}{n}\int_{-\pi}^{0} x \, d\cos nx$$

$$= -\frac{1}{n\pi}\left\{x \ \cos nx\Big|_{-\pi}^{0} - \int_{-\pi}^{0} \cos nx \, dx\right\}$$

$$= -\frac{1}{n\pi}\left\{0 \ \cos 0 + \pi\cos(-n\pi) - \frac{1}{n}\sin nx\Big|_{-\pi}^{0}\right\}$$

$$= -\frac{1}{n\pi}\left\{0 + (-1)^n - \frac{1}{n}(\sin 0 - \sin\,(-n\pi))\right\}$$

$$= -\frac{1}{n\pi}\left\{(-1)^n - \frac{1}{n}(0 - 0)\right\} = -\frac{(-1)^n}{n\pi}$$

$$II = \frac{1}{\pi} \int_{-\pi}^{0} \pi \ \sin nx \, dx = -\frac{1}{\pi}\frac{\pi}{n} \ \cos nx \Big|_{-\pi}^{0}$$

$$= -\frac{1}{n}\{ \cos 0 - \cos(-n\pi)\} = \frac{1}{n}\{1 + (-1)^n\}$$

$$III = \frac{1}{n\pi} \int_{0}^{\pi} x \, d \cos nx \ = \frac{1}{n\pi}\left\{ x \cos nx \Big|_{0}^{\pi} - \int_{0}^{\pi} \cos nx \, dx \right\}$$

$$= \frac{1}{n\pi}\left\{ \pi \ \cos n\pi - 0 \cos 0 + \frac{1}{n} \sin nx \Big|_{0}^{\pi} \right\}$$

$$= \frac{1}{n\pi}\left\{ \pi(-1)^n - 0 + \frac{1}{n}(\sin n\pi - \sin 0) \right\}$$

$$= \frac{1}{n\pi}\left\{ \pi(-1)^n + \frac{1}{n}(0 - 0) \right\} = \frac{(-1)^n}{n\pi}$$

Sehingga:

$$b_n = I + II + III$$

$$= -\frac{(-1)^n}{n\pi} + \frac{1}{n}\{1 + (-1)^n\} + \frac{(-1)^n}{n\pi}$$

$$= \frac{1}{n}\{1 + (-1)^n\}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{\{1 + (-1)^n\}}{n} \sin nx$$

4. Tentukan Deret fourier dari :

$$f(x) = \begin{cases} 0; \ -\pi < x < 0 \\ \sin x \, ; 0 < x < \pi \end{cases}$$

Penyelesaian

Menentukan Koefisien:

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x \ dx = -\frac{1}{\pi} \cos x\Big|_0^\pi$$

$$= \frac{1}{\pi} \{\cos\pi - \cos 0\} = \frac{\pi}{2}\{-1 - 1\} = -\pi$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx \ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left\{\frac{e^{ix} - e^{-ix}}{2i}\right\}\left\{\frac{e^{inx} + e^{-inx}}{2}\right\} dx = \frac{1}{4i\pi}\int_0^{\pi} \{e^{ix(1+n)} + e^{ix(1-n)} - e^{-ix(1-n)} - e^{-ix(1+n)}\} dx$$

$$= \frac{1}{4i\pi}\left\{\int_0^{\pi} e^{ix(1+n)} dx + \int_0^{\pi} e^{ix(1-n)} dx - \int_0^{\pi} e^{-ix(1+n)} dx - \int_0^{\pi} e^{-ix(1+n)} dx\right\}$$

$$= \frac{1}{4i\pi}\left\{\frac{e^{ix(1+n)}}{i(1+n)} + \frac{e^{ix(1-n)}}{i(1-n)} + \frac{e^{-ix(1-n)}}{i(1-n)} + \frac{e^{-ix(1+n)}}{i(1+n)}\right\}\Bigg|_0^{\pi}$$

$$= \frac{1}{4i\pi}\left\{\frac{e^{ix(1+n)}}{i(1+n)} + \frac{e^{-ix(1+n)}}{i(1+n)}\right\} + \left\{\frac{e^{ix(1-n)}}{i(1-n)} + \frac{e^{-ix(1-n)}}{i(1-n)}\right\}\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{(1+n)}\left[\frac{e^{ix(1+n)} + e^{-ix(1+n)}}{2}\right]\right\} + \left\{\frac{1}{(1-n)}\left[\frac{e^{ix(1-n)} + e^{-ix(1-n)}}{2}\right]\right\}\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{(1+n)}\cos\{x(1+n)\}\right\} + \left\{\frac{1}{(1-n)}\cos\{x(1-n)\}\right\}\right]\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{(1+n)}\cos\{\pi(1+n)\} - \cos 0\right\} + \left\{\frac{1}{(1-n)}\cos\{\pi(1-n)\} - \cos 0\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{(1+n)}\cos\{\pi(1+n)\} - 1\right\} + \left\{\frac{1}{(1-n)}\cos\{\pi(1-n)\} - 1\right\}\right]$$

$$= \frac{1}{2\pi}\left[2 - \frac{1}{(1+n)}\cos\{\pi(1+n)\} - \frac{1}{(1-n)}\cos\{\pi(1-n)\}\right]$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \; dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left\{\frac{e^{ix} - e^{-ix}}{2i}\right\}\left\{\frac{e^{inx} - e^{-inx}}{2i}\right\} dx = -\frac{1}{4\pi}\int_0^{\pi} \left\{e^{ix(1+n)} - e^{ix(1-n)} - e^{-ix(1-n)} + e^{-ix(1+n)}\right\} dx$$

$$= -\frac{1}{4\pi}\left\{\int_0^{\pi} e^{ix(1+n)} dx - \int_0^{\pi} e^{ix(1-n)} dx - \int_0^{\pi} e^{-ix(1+n)} dx + \int_0^{\pi} e^{-ix(1+n)} dx\right\}$$

$$= -\frac{1}{4\pi}\left\{\frac{e^{ix(1+n)}}{i(1+n)} - \frac{e^{ix(1-n)}}{i(1-n)} + \frac{e^{-ix(1-n)}}{i(1-n)} - \frac{e^{-ix(1+n)}}{i(1+n)}\right\}\Bigg|_0^{\pi}$$

$$= -\frac{1}{4\pi}\left\{\frac{e^{ix(1+n)}}{i(1+n)} + \frac{e^{-ix(1+n)}}{i(1+n)}\right\} - \left\{\frac{e^{ix(1-n)}}{i(1-n)} + \frac{e^{-ix(1-n)}}{i(1-n)}\right\}\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left\{\frac{1}{i(1+n)}\left[\frac{e^{ix(1+n)} + e^{-ix(1+n)}}{2}\right]\right\} - \left\{\frac{1}{i(1-n)}\left[\frac{e^{ix(1-n)} + e^{-ix(1-n)}}{2}\right]\right\}\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{i(1+n)}\cos\{x(1+n)\}\right\} - \left\{\frac{1}{i(1-n)}\cos\{x(1-n)\}\right\}\right]\Bigg|_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{i(1+n)}\cos\{\pi(1+n)\} - \cos 0\right\} - \left\{\frac{1}{i(1-n)}\cos\{\pi(1-n)\} - \cos 0\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\left\{\frac{1}{i(1+n)}\cos\{\pi(1+n)\} - 1\right\} - \left\{\frac{1}{i(1-n)}\cos\{\pi(1-n)\} - 1\right\}\right]$$

$$= \frac{1}{2\pi}\left[2 - \frac{1}{i(1+n)}\cos\{\pi(1+n)\} + \frac{1}{i(1-n)}\cos\{\pi(1-n)\}\right]$$

Sehingga:

$$f(x) = -\frac{\pi}{2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\left[2 - \frac{1}{(1+n)}\cos\{\pi(1+n)\} - \frac{1}{(1-n)}\cos\{\pi(1-n)\}\right]\cos nx + \left[2 - \frac{1}{i(1+n)}\cos\{\pi(1+n)\} + \frac{1}{i(1-n)}\cos\{\pi(1-n)\}\right]\sin nx$$