Some Application of IVP’s with 1st-Order ODE’s

Many dynamical phenomena in the reality can be explained clearly by using mathematical model. A lot of the model that described the dynamical are in the form of IVP’s with first-order ordinary differential equations. Some real problems will be discussed are:

1. Dynamics Population
2. Personal Finance
3. Molecularity of Chemical Reaction
4. Electrical Circuit (Resistor – Inductor)

IVP's: Dynamics of Population

Assumption:

the change of population is only influenced by births and deaths

Let

- **$\beta(t)$** is amount of births per population unit in unit of time (rate of birth)
- **$\delta(t)$** is amount of deaths per population unit in unit of time (rate of death)

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Then number of births and deaths for time interval $[t\ ,\ t + \Delta t]$ are given (approximation) by:

Births : $\beta(t)\ P(t)\ \Delta t$ and

Deaths : $\delta(t)\ P(t)\ \Delta t$,

where $P(t)$ is number of population at time **t**

Therefore, the change of population for interval of time s $[t\ ,\ t + \Delta t]$ is

$$\Delta P = \text{kelahiran} - \text{kematian}$$

$$\approx \beta(t)\ P(t)\ \Delta t - \delta(t)\ P(t)\ \Delta t$$

so that

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$$\frac{\Delta P}{\Delta t} \approx \left(\beta(t) - \delta(t)\right) P(t). \qquad 1.9$$

So, if $\Delta t \to 0$ then the error of approximation approches **zero** so that

$$\frac{dP(t)}{dt} = \left(\beta(t) - \delta(t)\right) P(t) \qquad 1.10$$

Equation of (**1.10**) is called the **general population equation**

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General solution of Eq. (**1.10**) is

$$P(t) = P(t_0)\, Exp\left(\int (\beta(t) - \delta(t))\, dt \right), \qquad 1.11$$

where $P(t_0)$ is number of population on initial obser-vation. If $\beta(t)$ and $\delta(t)$ are a contant functions, then Eq. (1.11) is to be

$$\boldsymbol{P(t) = P(t_0)\; e^{k\,t}}\,, \qquad 1.12$$

where $k = \beta - \delta$.

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Example 1.7: Let population number of alligator on initial observation ($t_0 = 0$) and rate of deaths **100** and **0**, respectively. If the rate of births of alligator is **0,0005 P(t)**, find

a. number of alligator on time t (in years)

b. when the number of alligator to be twice of initial population, and

c. draw some of integral curves.

Answer:

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Answer Ex. 1.7:

a) Known $P(t_0 = 0) = 100$, $\delta = 0$, and $\beta = 0{,}0005\, P(t)$, so that it goes to IVP as follows

$$\frac{dP}{dt} = (0{,}0005)\, P^2(t)\ , \qquad P(t_0) = 100. \qquad (*)$$

By using separation method, the general solution of $(*)$ is

$$\int \frac{1}{P^2}\, dP = \int (0{,}0005)\, dt \;\Leftrightarrow\; P(t) = \frac{2000}{C - t}. \qquad (*\,*)$$

At $t = 0$ it finds $P(0) = 100$ so that $C = 20$ and Eq. $(*\,*)$ becomes

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Ans. Ex. 1.7 (continued ....):

$$P(t) = \frac{2000}{20 - t}. \qquad (***)$$

b) Let t_1 be time needed alligator to be twice of initial population. Then $P(t_1) = 2\ P(0) = 200 = 2000/(20 - t_1)$ su that $t_1 = 10$. Thus, after 10 years alligator is to be twice of initial population.

c) Integral curves for some initial conditions is as follows:

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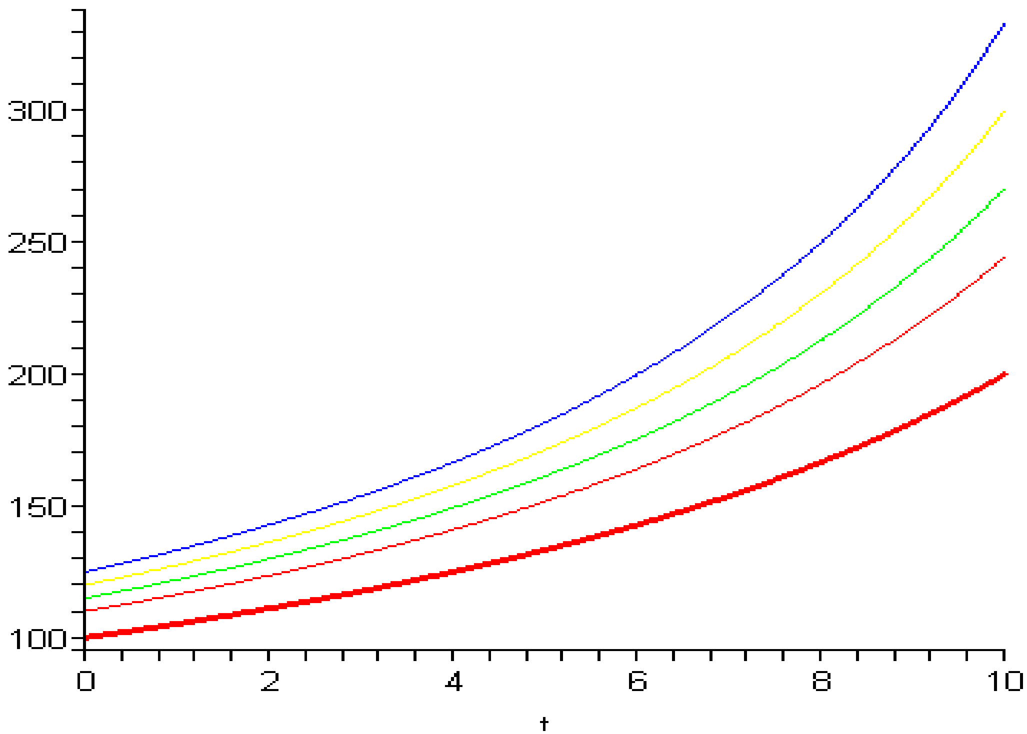
Ans. Ex. 1.7 (continued ....):



Integral Curves for IC’s: 100, 110, 115, 120, and 125

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Let $\beta(t)$ be a linear decreasing function with respect to $P(t)$ so that $\beta(t) = \beta_0 - \beta_1 P(t)$, where β_0 dan β_1 are both positive, and $\delta(t) = \delta_0$ be constant, then Eq. (**1.10**) becomes

$$\frac{dP(t)}{dt} = aP(t) - bP^2(t), \qquad (1.13)$$

where $a = \beta_0 - \delta_0$ and $b = \beta_1$. If both **a** and **b** are positive then Eq. (**1.13**) is called **Logistic Equation**. This equation is intruduced by the Belgian Mathematician and Demographer Pierre Francois Verhulst (1840).

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To see the behaviour of population clearly, it is important to rewrite Eq. (**1.13**) in the form:

$$\frac{dP(t)}{dt} = kP(t)(M - P(t)), \qquad \textbf{1.14}$$

where k = b and M = a/b. It is obvius that P(t) = M is a solution of (1.14). To have solution P(t) $\neq$ M, the method of separation variables can be applied and we have

$$P(t) = \frac{M}{1 + A\exp(-\mathrm{kMt})}, \qquad \textbf{1.15}$$

where A is integration constant.

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It is clear that P(t) = M can be expressed as (1.15) by setting A = 0 so that (1.15) is general solution of equation. If $P(t_0 = 0) = P_0$, it follows $A = (M - P_0)/P_o$ so that (**1.15**) becomes

$$P(t) = \frac{M\,P_0}{P_0 + (M - P_0)\exp(-kMt)}. \qquad \textbf{(1.16)}$$

So, the solution of IVP for Logistic Eq. Is (1.16) and P(t) appraoches to M as t tends to infinity. It means that the solution of Logistic Equation is bounded.

IVP’s: Dynamics of Population

Example 1.8: Suppose that in 1885 the population of certain country was 50 million and was growing at the rate of 750.000 people/year at that time. Suppose also that in 1940 its population was 100 million and was then growing at the rate of 1 million/year. Assume that this population satisfies the logistic equation. Determine both limiting population M and the predicted population for the year 2020.

Answer:

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Answer Ex. 1.8:

a) Known $P(t_0 = -55) = 50$ million, $dP(t)/dt|t=0$ = 0,75 million/thn, $P(t = 0) = 100$ million, and $dP(0)/dt$ = 1 million/year, so that we find the following system

$$0{,}75 = k(M - 50)\,50\ , \qquad 1 = k(M - 100)\,100. \qquad (*)$$

By solving that system, it finds M = 200 and $k = 10^{-10}$.
So, limit of population is 200 million and by setting t = 0 for 1940: